STUDENT'S NAME: ________________________________

TEACHER'S NAME: ________________________________

2023

HURLSTONE AGRICULTURAL HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE ASSESSMENT TASK 4

TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black or blue pen.
- NESA approved calculators may be used.
- A reference sheet is provided at the end of this question booklet.
- For questions in Section II, show all relevant mathematical reasoning and/or calculations.
- This examination paper is not to be removed from the examination centre.

Total marks: 100

Section I – 10 marks (pages 2 – 6)

- Attempt Questions 1 – 10. The multiple-choice answer sheet has been provided at the end of this question booklet.
- Allow about 15 minutes for this section.

Section II – 90 marks (pages 7 – 14)

- Attempt Questions 11 – 16, write your solutions in the answer booklets provided. Extra working pages are available if required.
- Allow about 2 hours and 45 minutes for this section.

Disclaimer: Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2023 HSC Mathematics Extension 2 Examination.

Section 1

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1. Which of the following is equivalent to $2e^{\frac{5\pi i}{6}}$?

   A. $\sqrt{3} - i$

   B. $\sqrt{3} + i$

   C. $-\sqrt{3} - i$

   D. $-\sqrt{3} + i$

2. Imagine ω is an imaginary cube root of unity, then what is $(1 + \omega - \omega^2)^{2020}$ equal to?

   A. $-2^{2020}\omega$

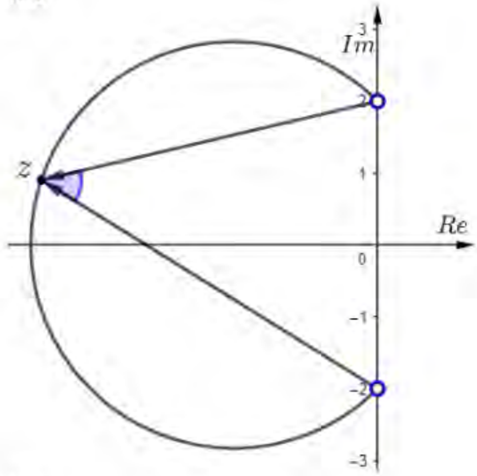
   B. $2^{2020}\omega$

   C. $-2^{2020}\omega^2$

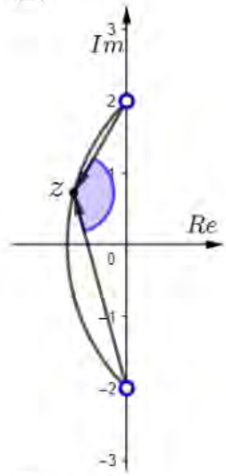
   D. $2^{2020}\omega^2$

3. Which diagram represents z such that $arg(\frac{z+2i}{z-2i}) = \frac{3\pi}{4}$?

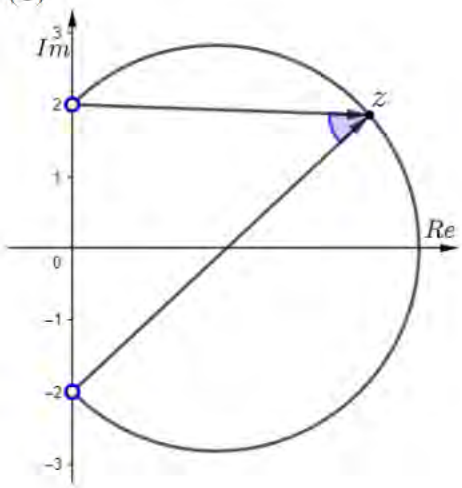
(A)



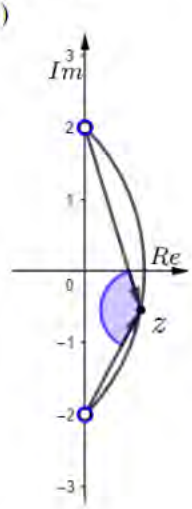
(B)



(C)



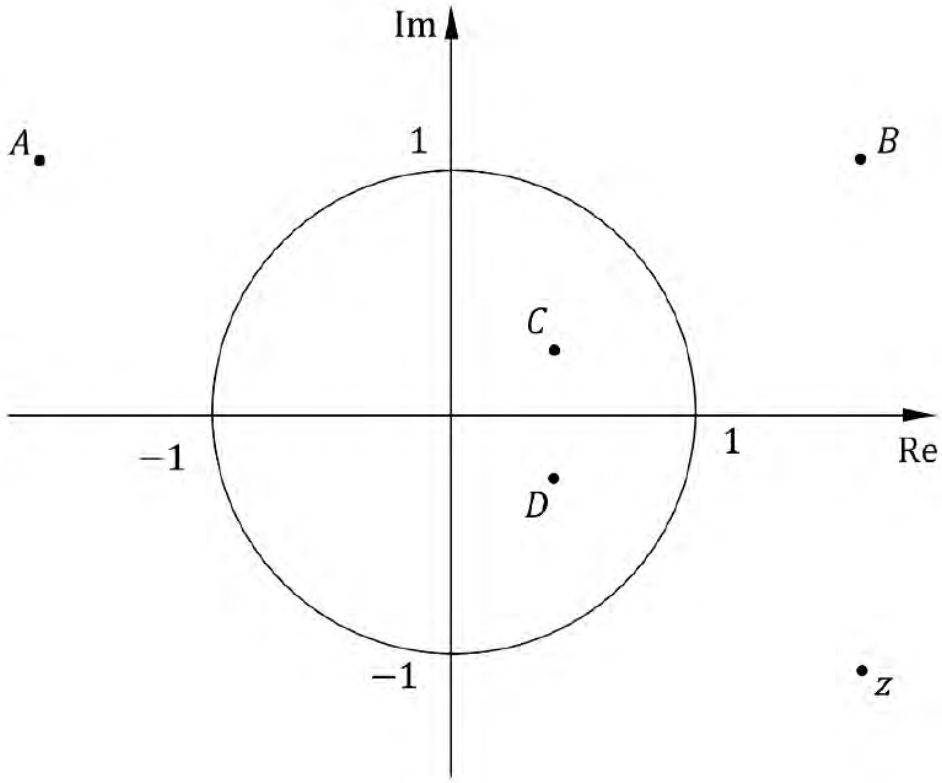
(D)



4. The diagram shows the complex number z in the fourth quadrant of the complex plane.

The modulus of z is 2.

Which of the points marked A, B, C or D best shows the position of $\frac{1}{z}$?



A. Point *A*

B. Point *B*

C. Point *C*

D. Point *D*

5. $OABC$ is a rectangle with $\overrightarrow{OA} = 3\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ and $\overrightarrow{OC} = 6\boldsymbol{i} + 4\boldsymbol{j} + a\boldsymbol{k}$ for some constant a.

What is the value of a?

A. –2
B. –3
C. –5
D. –6

6. The vector equation of a sphere is given by $\left| \underset{\sim}{v} - \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \right| = 52$.

Where does the point with position vector $\begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$ lie with respect to the sphere?

A. At the centre of the sphere.
B. Within the sphere, but not at the centre.
C. On the surface of the sphere.
D. Outside of the sphere.

7. Which of the following is **false**?

A. $\int_{-3}^{3} x^3\, e^{-x^2} dx = 0$
B. $\int_{-4}^{4} \frac{x^2}{x^2+4} dx = 2\int_{0}^{4} \frac{x^2}{x^2+4} dx$
C. $\int_{0}^{\pi} \sin^4\theta\; d\theta > \int_{0}^{\pi} \sin 4\theta\; d\theta$
D. $\int_{0}^{1} x^4\, dx < \int_{0}^{1} x^5\, dx$

8. Which of the following uses a correct substitution for $\int_0^{\sqrt{3}} \frac{\ln(\tan^{-1} x)}{1+x^2} dx$?

A. $\int_0^{\frac{\pi}{3}} \ln u \, du$

B. $\int_0^{\frac{\pi}{3}} \frac{\ln u}{1+\tan^2 u} du$

C. $\int_0^{\sqrt{3}} \ln u \, du$

D. $\int_0^{\sqrt{3}} \frac{\ln u}{1+\tan^2 u} du$

9. Which of the following statements is true?

A. An example is enough to prove a “for all” (∀) statement.

B. An example is enough to disprove a “there exists” (∃) statement.

C. A counter-example is enough to disprove a “for all” (∀) statement.

D. A counter-example is enough to disprove a “there exists” (∃) statement.

10. Consider the statement: “If I pass the exam, then you will pass the exam.”

Which of the following is logically equivalent to this statement?

A. If I do not pass the exam, then you will not pass the exam.

B. If you pass the exam, then I will pass the exam.

C. If you do not pass the exam, then I will not pass the exam.

D. You will pass the exam only if I pass the exam.

END OF SECTION I

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section

Answer the questions in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

MARKS

(a) Let $z = 2 - 3i$ and $w = -1 + i$.

Find $z^2 - \overline{\omega}$ in the form $x + iy$ where x and y are real numbers. **2**

(b) The complex numbers $1 + 3i$ and $4 + 2i$ are denoted by u and v respectively

(i) Find $\frac{u}{v}$ in the form $x + iy$. **2**

(ii) Hence, justify why $\tan^{-1} 3 - \tan^{-1} \frac{1}{2} = \frac{\pi}{4}$. **1**

(c) Consider $f(x) = x^4 + 2x^3 + 2x^2 + 26x + 169$.

Given $f(2 + 3i) = 0$, fully factorise $f(x)$ over the set of complex numbers. **3**

(d) Sketch on an Argand diagram $|z + 3i| = |z - i|$ **1**

(e) Let $z^2 = 4 + 4i$

(i) If $z^2 w = z^2 - \overline{z^2}\sqrt{3}$, find w and express it in modulus-argument form. **2**

(ii) Find all solutions for Arg(z). **2**

(iii) If iz is a solution to $x^n + k = 0$, where k and n are positive integers,
Find a possible value of n. **2**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet.

MARKS

(a) (i) By solving the equation $z^3 + 1 = 0$, find the three cube roots of -1. **2**

(ii) Let ω be a non-real cube root of -1. Show that $\omega^2 = \omega - 1$. **1**

(iii) Hence simplify $(1-\omega)^6$ **2**

(b) By first writing $\sqrt{3} - i$ in exponential form, find the roots of the equation:

$$z^4 = \sqrt{3} - i.$$

Give your answers in exponential form using principal arguments. **3**

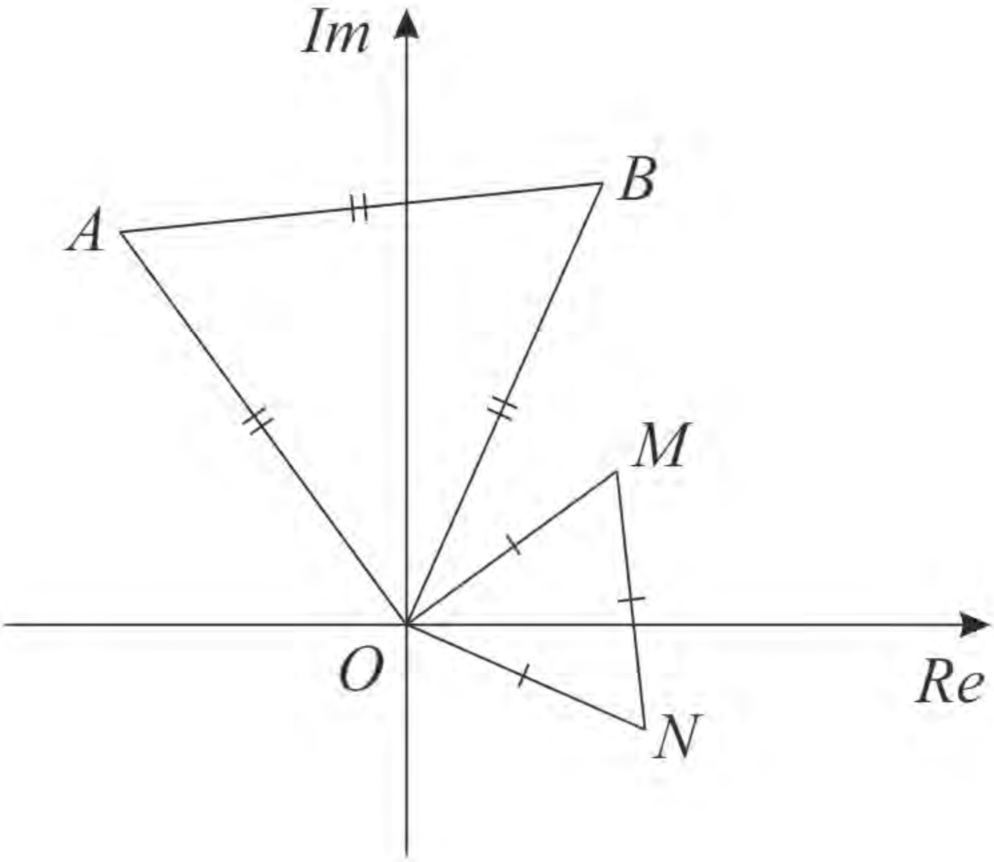
(c) If $z = 6e^{i\frac{\pi}{3}}$, simplify z^4. Give your answer in modulus and (principal) argument form. **2**

(d) Evaluate:

$$1 + e^{\frac{2\pi i}{3}} + \left(e^{\frac{2\pi i}{3}}\right)^2 + \left(e^{\frac{2\pi i}{3}}\right)^3 + \cdots + \left(e^{\frac{2\pi i}{3}}\right)^{3n}.$$ **2**

Question 12 continues on the next page.

(e)



The diagram above shows the points *O*, *A*, *B*, *M* and *N* on the complex plane.

These points correspond to the complex numbers *0*, *a*, *b*, *m* and *n* respectively.

The triangles *OAB* and *OMN* are equilateral. Let $\alpha = e^{\frac{i\pi}{3}}$.

(i) Explain why $m = \alpha n$. **1**

(ii) Show that *AM* = *BN*. **2**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet.

MARKS

(a) Let $\underset{\sim}{u} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} a \\ 2 \\ -1 \end{pmatrix}$.

Given that the vector projection of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{u}$ is $\begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ -\frac{4}{9} \end{pmatrix}$, find the value of a. **2**

(b) The points A, B, C are collinear where $\overrightarrow{OA} = \boldsymbol{i} - \boldsymbol{j}$, $\overrightarrow{OB} = -3\boldsymbol{i} - \boldsymbol{k}$ and $\overrightarrow{OC} = 2\boldsymbol{i} + a\boldsymbol{j} + b\boldsymbol{k}$ for some constants a and b.

What are the values of a and b? **3**

(c) A sphere S_1 with centre $C(-3, -5, 10)$ passes through the point with coordinates $A(3, -3, 6)$.

(i) Show that the vector equation of S_1 is $\left| \underset{\sim}{u} - \begin{pmatrix} -3 \\ -5 \\ 10 \end{pmatrix} \right| = 2\sqrt{14}$. **1**

(ii) Write down the Cartesian equation of S_1. **1**

(iii) The vector equation of another sphere of S_2 is $\left| \underset{\sim}{r} - \begin{pmatrix} -9 \\ 4 \\ 7 \end{pmatrix} \right| = \sqrt{14}$

Prove that the two spheres S_1 and S_2 touch each other at a single point. **2**

(iv) The vector equation of the line m is given as

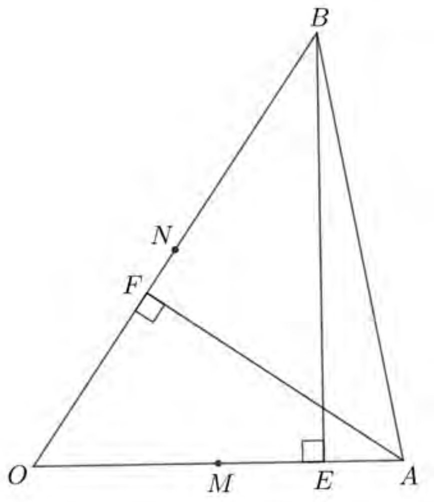
$$\underset{\sim}{v} = \begin{pmatrix} -6 \\ -3 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad \text{where } \lambda \in \mathbb{R}.$$

Find the value(s) of λ where the line m intersects the sphere S_1. **3**

Question 13 continues on the next page.

(d) In ΔOAB below, *BE* is the altitude from *B* to *OA*, and *AF* is the altitude from *A* to *OB*.

$\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$.



Given that *M*, *N* are the midpoints of *OA*, *OB* respectively,

use vector methods to show that $|\overrightarrow{OM}| \times |\overrightarrow{OE}| = |\overrightarrow{ON}| \times |\overrightarrow{OF}|$. **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet.

MARKS

(a) Find $\int \frac{dx}{x^2+6x+13}$ **2**

(b) Use integration by parts to find $\int x3^x dx$. **3**

(c) (i) Prove that $\sqrt{x}\left(1-\sqrt{x}\right)^{n-1} = \left(1-\sqrt{x}\right)^{n-1} - \left(1-\sqrt{x}\right)^{n}$. **1**

(ii) Let $I_n = \int_0^1 \left(1-\sqrt{x}\right)^n dx$ where n = 1, 2, 3, …

Show that $I_n = \frac{n}{n+2} I_{n-1}$. **3**

(iii) Hence, evaluate I_{2023} **2**

(d) Use the substitution $u = \cos 2\theta$ to evaluate $\int_{\frac{1}{2}}^{1} \sqrt{\frac{1-u}{1+u}}\, du$. **4**

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet.

MARKS

(a) Prove that the statement: $\exists\ a, b, c \in \mathbb{Z}$ such that $a^2 + b^3 = c$ where a, b, c, are odd integers, is FALSE. **2**

(b) Prove that a 3-digit integer is divisible by 9 **if and only if**

the sum of all its digits is ALSO divisible by 9. **3**

(c) (i) Prove that $(a+b)^2 \geq 4ab$ where $a, b \in \mathbb{R}$. **1**

(ii) Hence, prove that $\left(x^2 + 3x + 2 + \frac{1}{x+1}\right)^2 \geq 4x$, where $x \in \mathbb{R}$ and $x \neq -1$. **2**

(d) Prove that $\log_x y$ is irrational if x is even, y is odd and $x, y \in \mathbb{Z}^+$. **2**

(e) Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n}{dx^n}\left((x+1)e^{x-1}\right) = (x+n+1)e^{x-1}.$$ **2**

(f) Use mathematical induction to prove: $e^{-x} < \frac{1}{x}$ for all $x > 0, x \in \mathbb{Z}$. **3**

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet.

MARKS

(a) Let $P(x, y)$ represent the complex number *z* on the Argand Diagram.

Let Q be the point $(-4,8)$.

(i) Sketch the graph of the solution to: $|\overrightarrow{QP}| = |z|$
Include labels of the points where the solution crosses the *x* and *y* axes. **1**

(ii) Hence or otherwise, solve: $z + 4 - 8i = |z|$ **2**

(b) The velocity of a body moving in simple harmonic motion along the *x*-axis

is given by: $v^2 = 21 - 4x - x^2$.

Find the amplitude of this motion. **1**

(c) A particle moves in a straight line so that its position at any value, *t*, in seconds $(t \geq 0)$

is given by: $x = 5\cos\left(3t + \frac{\pi}{4}\right)$.

(i) Show that the motion of the particle is simple harmonic motion,
and hence state the period of this motion **2**

(ii) Find the first 2 times that the particle reaches its greatest speed.

Give your answers correct to three decimal places. **2**

(d) Let $z = e^{i\theta}$.

(i) Show that $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$. **2**

(ii) Show that $\left(z - \frac{1}{z}\right)^5 = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$. **2**

(iii) Hence find $\int \sin^5\theta\, d\theta$. **3**

End of Question 16

End of examination

2022 Trial Higher School Certificate Examination

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, r \neq 1$$

$$S = \frac{a}{1-r}, |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

$A = \dfrac{1}{2}ab\sin C$

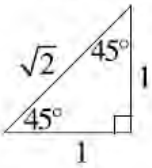
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

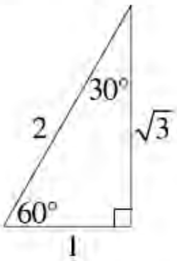
$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$





Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1+t^2}$

$\cos A = \dfrac{1-t^2}{1+t^2}$

$\tan A = \dfrac{2t}{1-t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

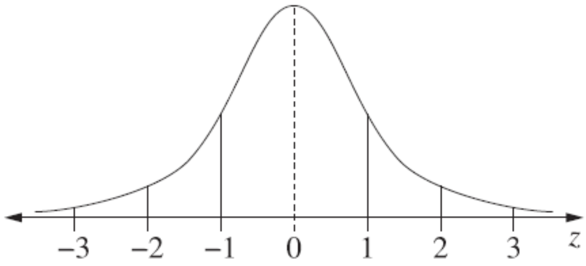
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \le x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^{r} + \cdots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2}+y^{2}}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^{n} &= r^{n}(\cos n\theta + i\sin n\theta) \\ &= r^{n}e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$

HURLSTONE AGRICULTURAL HIGH SCHOOL

2023 Trial Higher School Certificate Examination
Mathematics Extension 2

Name ____________________ Teacher ____________________

Section I – Multiple Choice Answer Sheet

Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

A ⊗ B ⊗ (correct) C ○ D ○

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

Hurlstone Agricultural High School

Mathematics Extension 2 Trial HSC 2023 Marking Guidelines

Multiple Choice

Q1. All responses have a modulus equal to 2, so we need to differentiate between the arguments, noting that $e^{\frac{5\pi i}{6}}$ lies in the second quadrant. There is only one candidate.

Answer: D

Q2. For cube roots of unity, $1 + \omega + \omega^2 = 0$ so $1 + \omega - \omega^2 = -2\omega^2$.

$$(-2\omega^2)^{2020} = 2^{2020}\omega^{4040}$$

Since the digits of 4040 add to 8, then there is a remainder of 2 when dividing by 3.

$$w^{4040} = \omega^2$$

Answer: D

Q3. $arg(z + 2i) - arg(z - 2i) = \frac{3\pi}{4}$.

Hence the vector from 2*i* to *z* is an anti-clockwise rotation of $\frac{3\pi}{4}$ from the vector from 2*i* to *z*.

Answer: D

Q4. $\frac{1}{z}$ will have the argument of *z* multiplied by negative 1, and it's modulus will equal $\frac{1}{2}$.

Answer: C

Q5. $\overrightarrow{OA} \perp \overrightarrow{OC}$, so $\overrightarrow{OA} \cdot \overrightarrow{OC} = 0$

$$3 \times 6 - 2 \times 4 + 2a = 0$$

$$a = -5$$

$\therefore$C

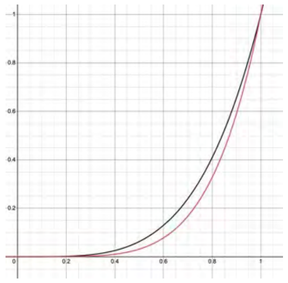
Q6. distance $= \sqrt{(3+3)^2 + 0^2 + (-2-2)^2}$

$$= \sqrt{52} < 52$$

$\therefore$ point is inside sphere (but not at centre)

$\therefore$B

Q7. $y = x^4$ is above $y = x^5$ in the domain $0 \leq x \leq 1$,



$\therefore$ D

Note: A is odd function $\therefore$ true

B is even function $\therefore$ true

C $y = sin^4\theta$ is entirely above x-axis

$y = sin4\theta$ has area above and below the x-axis

so $y = sin^4\theta$ integral is greater

Q8. $u = tan^{-1}x$ $\quad x = 0 \Rightarrow u = 0$

$du = \frac{1}{1+x^2}dx$ $\quad x = \sqrt{3} \Rightarrow u = \frac{\pi}{3}$

Fits with A $\therefore$ A

Q9**:**. A "for all" statement implies that a counter-example does not exist.

Hence the counter-example disproves it.

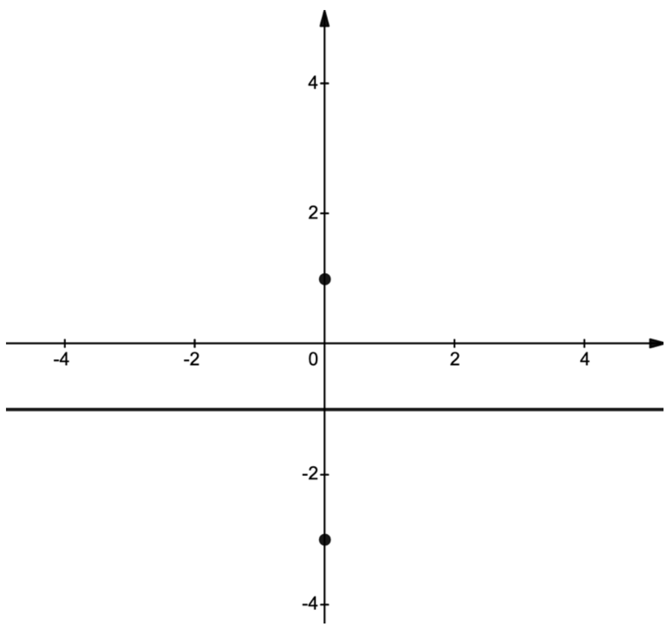
Answer: C

Q10: This is the contra-positive to the statement in the question,

which is therefore equivalent.

Answer: C

2023 Yr12 HSC Assessment Task 4	
Question 11 Solutions and Marking Guidelines	
Outcomes Addressed in this Question	
MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems	
Solutions	**Marking Guidelines**
a) $$\begin{aligned} z^2 - \bar{w} &= (2-3i)^2 - (-1-i) \\ &= 4 - 12i - 9 + 1 + i \\ &= -4 - 11i \end{aligned}$$	2 marks Correct solution 1 mark Substantial progress to correct solution
b i) $$\frac{u}{v} = \frac{1+3i}{4+2i} \times \frac{4-2i}{4-2i}$$ $$= \frac{4+12i-2i+6}{16+4}$$ $$= \frac{10+10i}{20}$$ $$= \frac{1}{2} + \frac{1}{2}i$$	2 marks Correct solution in $x + yi$ form 1 mark Substantial progress to correct solution
ii) $$\tan^{-1} 3 = \text{mod}(u)$$ $$\tan^{-1} \frac{1}{2} = \text{mod}(v)$$ $$\text{mod}\left(\frac{1}{2} + \frac{1}{2}i\right) = \frac{\pi}{4} = \text{mod}\left(\frac{u}{v}\right)$$ $$\tan^{-1} 3 - \tan^{-1} \frac{1}{2} = \text{mod}(u) - \text{mod}(v)$$ $$= mod\left(\frac{u}{v}\right)$$ $$= \frac{\pi}{4}$$	1 mark Correct Soltuon

c) if $f(2+3i)=0$, then $f(2-3i)=0$ as complex roots come in conjugate pairs for polynomials with real coefficients. So $f(x)=(x-(2+3i))(x-(2-3i))(x^2+bx+c)$ $=(x^2-4x+13)(x^2+bx+c)$ $13c=169$ so $c=13$ $13bx-52x=26x$ $b=6$ Factorising $x^2+6x+13$ gives $(x-(-3+2i))(x-(-3-2i))$ Hence $f(x)=(x-(2+3i))(x-(2-3i))(x+(3+2i)(x+(3-2i))$	3 marks Correct solution with clear reasoning 2 marks Substantial progress towards solution 1 mark Some progress towards solution.
d) $\|z+3i\|=\|z-i\|$ $\sqrt{x^2+(y+3)^2}=\sqrt{x^2+(y-1)^2}$ $(y+3)^2=(y-1)^2$ $0=-8y-8$ $y=-1$ 	1 mark Correct graph.

e i) $$z^2 = 4+4i$$ $$z^2 w = z^2 - \bar{z}\sqrt{3}$$ $$(4+4i)w = 4+4i-(4-4i)\sqrt{3}$$ $$w = 1-\frac{(4-4i)\sqrt{3}}{(4+4i)}\times\frac{(4-4i)}{(4-4i)}$$ $$= 1-\frac{(16-32i-16)\sqrt{3}}{16+16}$$ $$= 1+i\sqrt{3}$$ $$= 2\cos\frac{\pi}{3}+2i\sin\frac{\pi}{3}$$	2 marks Correct solution in mod-arg form 1 mark Substantial progress to correct solution
ii) $$\arg(z^2) = 2\arg(z)$$ $$\tan^{-1}\frac{4}{4} = 2\arg(z)$$ $$\frac{\pi}{4} = 2\arg(z)$$ $$\arg(z) = \frac{\pi}{8}$$	2 marks Correct solution 1 mark Substantial progress to correct solution
iii) $$\arg(z) = \frac{\pi}{8} \text{ and } \arg(iz) = \frac{\pi}{8}+\frac{\pi}{2} = \frac{5\pi}{8}$$ $$x^n + k = 0$$ $$x^n = -k$$ $$\arg((\mathrm{iz})^n) = \arg(-\mathrm{k})$$ $$\frac{5\pi}{8}n = 2\pi+\pi j \quad \text{where } j \in \mathbb{Z}$$ $n = 8$ is one solution	2 marks Correct solution 1 mark Some progress to correct solution

2023 Yr12 HSC Assessment Task 4

Question 12 Solutions and Marking Guidelines

Outcomes Addressed in this Question

MEX12-4
uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

Solutions	**Marking Guidelines**
a i) $$z^3+1=0$$ $$z^3=-1$$ $$z=-1, e^{\frac{\pi}{3}i}, e^{-\frac{\pi}{3}i}$$	2 marks Correct solution 1 mark Substantial progress to correct solution
ii) $$z^3+1=0$$ $$(z+1)(z^2-z+1)=0$$ Since $z \neq -1$ (w is a non-real root) then $w^2-w+1=0$ Hence $w^2=w-1$	1 mark Correct solution
iii) $$(1-w)^6=\left(-(w-1)\right)^6$$ $$=\left(-(w^2)\right)^6$$ $$=w^{12}$$ $$=1$$	2 marks Correct fully simplified solution 1 mark Substantial progress to correct solution
b) $$\text{mod}\left(\sqrt{3}-i\right)=2$$ $$\arg\left(\sqrt{3}-i\right)=\tan^{-1}\frac{1}{\sqrt{3}}=-\frac{\pi}{6}$$ $$z^4=2e^{-\frac{\pi}{6}i}$$ $$z=\sqrt[4]{2}e^{-\frac{\pi}{24}i}$$ Roots are: $\sqrt[4]{2}e^{-\frac{\pi}{24}i}$, $\sqrt[4]{2}e^{-\frac{13\pi}{24}i}$, $\sqrt[4]{2}e^{\frac{11\pi}{24}i}$, $\sqrt[4]{2}e^{\frac{23\pi}{24}i}$ (roots evenly spaced by $\frac{\pi}{2}$)	3 marks Correct solution with principle roots within $(-\pi,\pi]$ 2 marks Substantial progress towards solution 1 mark Some progress towards solution.

c) $$z = 6e^{\frac{\pi}{3}i}$$ $$z^4 = 6^4 e^{\frac{4\pi}{3}i}$$ $$= 6e^{-\frac{2\pi}{3}i}$$ $$= 6^4\left(\cos -\frac{2\pi}{3} + i\sin -\frac{2\pi}{3}\right)$$	2 marks Correct solution within $(-\pi, \pi]$ 1 mark Significant progress towards solution.
d) $1, e^{\frac{2\pi}{3}i}$ and $e^{\frac{4\pi}{3}i}$ are roots to $z^3 - 1 = 0$ We have $(z-1)(z^2+z+1) = 0$, hence $1 + e^{\frac{2\pi}{3}i} + e^{\frac{4\pi}{3}i} = 0$ This is repeated every three terms in the series. There are $3n + 1$ terms in the series. Hence, $0 + 0 + \cdots + 0 + \left(e^{\frac{2\pi}{3}i}\right)^{3n} = e^{2n\pi i} = 1$	2 marks Correct solution 1 mark Substantial progress to correct solution
e i) m and n are complex numbers with the same moduli and m is a rotation of n by $\frac{\pi}{3}$ (equilateral triangle angles). Hence $m = e^{\frac{\pi}{3}i}n$ $= \alpha n$	1 mark Correct solution
ii) $a = \alpha b$(same reasoning as part i)) $\overrightarrow{AM} = \overrightarrow{OA} - \overrightarrow{OM}$ $= \alpha\overrightarrow{OB} - \alpha\overrightarrow{ON}$ $= \alpha\left(\overrightarrow{OB} - \overrightarrow{ON}\right)$ $= \alpha\overrightarrow{BN}$ $\left\lvert\overrightarrow{AM}\right\rvert = \left\lvert\alpha\overrightarrow{BN}\right\rvert$ $= \lvert\alpha\rvert\left\lvert\overrightarrow{BN}\right\rvert$ $= 1\left\lvert\overrightarrow{BN}\right\rvert$ $\left\lvert\overrightarrow{AM}\right\rvert = \left\lvert\overrightarrow{BN}\right\rvert$	2 marks Correct solution with consistent notation 1 mark Significant progress to correct solution

Year 12	Mathematics Extension 2 2023	TASK 4 (TRIAL)
Question No. 13	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

MEX12-3 uses vectors to model and solve problems in two and three dimensions.

Part / Outcome	**Solutions**	**Marking Guidelines**
(a)	$\text{proj}_{\underset{\sim}{u}}\underset{\sim}{v} = \dfrac{\underset{\sim}{v}\cdot\underset{\sim}{u}}{\underset{\sim}{u}\cdot\underset{\sim}{u}}\times\underset{\sim}{u}$ $= \dfrac{2a-2+2}{4+1+4}\times\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ $\begin{pmatrix}\frac{4}{9}\\-\frac{2}{9}\\-\frac{4}{9}\end{pmatrix} = \dfrac{2a}{9}\times\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ $\dfrac{9}{2}\begin{pmatrix}\frac{4}{9}\\-\frac{2}{9}\\-\frac{4}{9}\end{pmatrix} = a\times\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ $\begin{pmatrix}2\\-1\\-2\end{pmatrix} = a\times\begin{pmatrix}2\\-1\\-2\end{pmatrix} \qquad \therefore a=1$	**2 marks** – Correct solution **1 mark** – Substantially correct
(b)	$\overrightarrow{AB} = -3\underset{\sim}{i}-\underset{\sim}{k}-\left(\underset{\sim}{i}-\underset{\sim}{j}\right) \qquad = -4\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}$ $\overrightarrow{BC} = 2\underset{\sim}{i}+a\underset{\sim}{j}+b\underset{\sim}{k}-\left(-3\underset{\sim}{i}-\underset{\sim}{k}\right) = 5\underset{\sim}{i}+a\underset{\sim}{j}+\left(b+1\right)\underset{\sim}{k}$ $\overrightarrow{BC} = \lambda\overrightarrow{AB}$ $5\underset{\sim}{i}+a\underset{\sim}{j}+\left(b+1\right)\underset{\sim}{k} = \lambda\left(-4\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}\right)$ $-4\lambda=5 \qquad a=\lambda \qquad b+1=-\lambda$ $\lambda=-\dfrac{5}{4}$ → $\quad =-\dfrac{5}{4} \qquad b=\dfrac{5}{4}-1=\dfrac{1}{4}$	**3 marks** – Correct solution **2 marks** – Substantially correct **1 mark** – Partial progress towards correct solution

Question 13 continued…

(c)(i)	radius $= \sqrt{(-3-3)^2+(-5+3)^2+(10-6)^2}$ $= \sqrt{56} = 2\sqrt{14}$ centre is $(-3,-5,10)$ $\therefore S_1$ is $\left\| \underset{\sim}{u} - \begin{pmatrix} -3 \\ -5 \\ 10 \end{pmatrix} \right\| = 2\sqrt{14}$	**1 mark** – correct solution
(c)(ii)	cartesian equation of S_1 is $(x+3)^2+(y+5)^2+(z-10)^2 = 56$	**1 mark** – correct equation
(c)(iii)	Distance between centres is $\sqrt{(-9+3)^2+(4+5)^2+(7-10)^2}$ $= \sqrt{126}$ $= 3\sqrt{14}$ $= \sqrt{14}+2\sqrt{14}$ (sum of the two radii) $\therefore$ S_1 & S_2 meet at a single point	**2 marks** – Correct solution **1 mark** – Substantially correct
(c)(iv)	Equate $\underset{\sim}{v}$ and $\underset{\sim}{u}$ (ie sub $\underset{\sim}{v}$ into S_1) $\left\| \begin{pmatrix} -6+2\lambda \\ -3+\lambda \\ 11+\lambda \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \\ 10 \end{pmatrix} \right\| = 2\sqrt{14}$ $\left\| \begin{pmatrix} -3+2\lambda \\ 2+\lambda \\ 1+\lambda \end{pmatrix} \right\| = 2\sqrt{14}$ $(-3+2\lambda)^2+(2+\lambda)^2+(1+\lambda)^2 = (2\sqrt{14})^2$ $9-12\lambda+4\lambda^2+4+4\lambda+\lambda^2+1+2\lambda+\lambda^2 = 56$ $6\lambda^2-6\lambda-42=0$ $\lambda^2-\lambda-7=0$ $\lambda = \dfrac{1\pm\sqrt{29}}{2}$	**3 marks** – Correct solution **2 marks** – Substantially correct **1 mark** – Partial progress towards correct solution

(d)	*Question 13 continued…* $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}\underset{\sim}{a}$ and $\overrightarrow{OE} = \lambda\overrightarrow{OA} = \lambda\underset{\sim}{a}$ $\overrightarrow{ON} = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\underset{\sim}{b}$ and $\overrightarrow{OF} = \mu\overrightarrow{OB} = \mu\underset{\sim}{b}$ $\left\lvert\overrightarrow{OM}\right\rvert\left\lvert\overrightarrow{OE}\right\rvert\cos\theta = \overrightarrow{OM}\cdot\overrightarrow{OE}$ $\left\lvert\overrightarrow{OM}\right\rvert\left\lvert\overrightarrow{OE}\right\rvert\cos 0 = \frac{1}{2}\underset{\sim}{a}\cdot\lambda\underset{\sim}{a}$ $\therefore \left\lvert\overrightarrow{OM}\right\rvert\left\lvert\overrightarrow{OE}\right\rvert = \frac{1}{2}\lambda\underset{\sim}{a}\cdot\underset{\sim}{a} \quad ...[1]$ similarly $\left\lvert\overrightarrow{ON}\right\rvert\left\lvert\overrightarrow{OB}\right\rvert = \frac{1}{2}\mu\underset{\sim}{b}\cdot\underset{\sim}{b} \quad ...[2]$ now, $\overrightarrow{OA}\perp\overrightarrow{BE}$ so, $\overrightarrow{OA}\cdot\overrightarrow{BE} = 0$ $\overrightarrow{OA}\cdot\left(\overrightarrow{OE}-\overrightarrow{OB}\right) = 0$ $a\cdot\left(\lambda\underset{\sim}{a}-\underset{\sim}{b}\right) = 0$ $\lambda\underset{\sim}{a}\cdot\underset{\sim}{a} = \underset{\sim}{a}\cdot\underset{\sim}{b} \quad ...[3]$ similarly, $\overrightarrow{OB}\perp\overrightarrow{AF}$ $\mu\underset{\sim}{b}\cdot\underset{\sim}{b} = \underset{\sim}{a}\cdot\underset{\sim}{b} \quad ...[4]$ $\therefore \quad \lambda\underset{\sim}{a}\cdot\underset{\sim}{a} = \mu\underset{\sim}{b}\cdot\underset{\sim}{b} \quad \left(\textit{from } [3] \,\&\, [4]\right)$ $\frac{1}{2}\lambda\underset{\sim}{a}\cdot\underset{\sim}{a} = \frac{1}{2}\mu\underset{\sim}{b}\cdot\underset{\sim}{b}$ $\therefore \left\lvert\overrightarrow{OM}\right\rvert\left\lvert\overrightarrow{OE}\right\rvert = \left\lvert\overrightarrow{ON}\right\rvert\left\lvert\overrightarrow{OF}\right\rvert \quad \left(\textit{from } [1] \,\&\, [2]\right)$	**3 marks –** Correct solution **2 marks –** Substantially correct solution **1 mark –** Partial (non-trivial) progress towards correct solution

Outcomes Addressed in this Question

MEX12-5 applies techniques of integration to structured and unstructured problems.

Part / Outcome	Solutions	Marking Guidelines
(a)	$\int \frac{dx}{x^2+6x+13} = \int \frac{dx}{x^2+6x+9+4}$ $= \int \frac{dx}{(x+3)^2+2^2}$ $= \frac{1}{2}\tan^{-1}\frac{x+3}{2}+c$	**2 marks** – Correct solution **1 mark** – Substantially correct (*Note, omitting the constant prevents access to full marks in this question. This is a stock-standard question, and detail is always important*)
(b)	$\int x3^x\,dx = uv - \int v\,du$ $u = x \quad dv = 3^x$ $du = dx \quad v = \frac{1}{\ln 3}3^x$ $= \frac{x3^x}{\ln 3} - \frac{1}{\ln 3}\int 3^x\,dx$ $= \frac{x3^x}{\ln 3} - \frac{1}{\ln 3}\times\frac{3^x}{\ln 3} + C$ $= \frac{3^x}{\ln 3}\left(x - \frac{1}{\ln 3}\right) + C$	**3 marks** – Correct solution **2 marks** – Substantially correct **1 mark** – Partial progress towards correct solution
(c)(i)	$\text{LHS} = \sqrt{x}\left(1-\sqrt{x}\right)^{n-1}$ $= \left(1-1+\sqrt{x}\right)\left(1-\sqrt{x}\right)^{n-1}$ $= \left(1-\left(1-\sqrt{x}\right)\right)\left(1-\sqrt{x}\right)^{n-1}$ $= \left(1-\sqrt{x}\right)^{n-1} - \left(1-\sqrt{x}\right)^{n}$ $= \text{RHS}$	**1 mark** – correct solution

	Question 14 continued…	
(c)(ii)	$$I_n = \int_0^1 \left(1-\sqrt{x}\right)^n dx$$ $$= \left[uv\right]_0^1 - \int_0^1 v\,du$$ $$= \left[x\left(1-\sqrt{x}\right)^n\right]_0^1 + \frac{n}{2}\int_0^1 \frac{\left(1-\sqrt{x}\right)^{n-1} x}{\sqrt{x}}\,dx$$ $$= 0 - 0 + \frac{n}{2}\int_0^1 \sqrt{x}\left(1-\sqrt{x}\right)^{n-1} dx$$ $$= \frac{n}{2}\int_0^1 \left[\left(1-\sqrt{x}\right)^{n-1} - \left(1-\sqrt{x}\right)^n\right] dx \quad \left(\textit{from } (i)\right)$$ $$I_n = \frac{n}{2}\int_0^1 \left(1-\sqrt{x}\right)^{n-1} dx - \frac{n}{2}\int_0^1 \left(1-\sqrt{x}\right)^n dx$$ $$= \frac{n}{2} I_{n-1} - \frac{n}{2} I_n$$ $$I_n + \frac{n}{2} I_n = \frac{n}{2} I_{n-1}$$ $$\frac{n+2}{2} I_n = \frac{n}{2} I_{n-1}$$ $$I_n = \frac{n}{n+2} I_{n-1}$$	**3 marks** – Correct solution **2 marks** – Substantially correct **1 mark** – Partial progress towards correct solution
(c)(iii)	$$I_1 = \int_0^1 \left(1-\sqrt{x}\right) dx$$ $$= \left[x - \frac{2x\sqrt{x}}{3}\right]_0^1$$ $$= 1 - \frac{2}{3} = \frac{1}{3}$$ $$I_n = \frac{n}{n+2} I_{n-1}$$ $$I_{2023} = \frac{2023}{2025} I_{2022}$$ $$= \frac{2023}{2025} \times \frac{2022}{2024} I_{2021}$$ $$= \frac{2023 \times 2022 \times 2021 ... \times 3 \times 2}{2025 \times 2024 \times 2023 ... \times 5 \times 4} \times I_1$$ $$= \frac{3 \times 2}{2025 \times 2024} \times \frac{1}{3}$$ $$= \frac{1}{2\ 047\ 276}$$	**2 marks** – Correct solution **1 mark** – Substantially correct

	Question 14 continued…	
(d)	$\int_{1/2}^{1}\sqrt{\frac{1-u}{1+u}}\,du$ $=\int_{\frac{\pi}{6}}^{0}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\times -2\sin 2\theta\, d\theta$ $=-2\int_{\frac{\pi}{6}}^{0}\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\times \sin 2\theta\, d\theta$ $=2\int_{0}^{\frac{\pi}{6}}\frac{\sin\theta}{\cos\theta}\times 2\sin\theta\cos\theta\, d\theta$ $=2\int_{0}^{\frac{\pi}{6}}2\sin^2\theta\, d\theta$ $=2\int_{0}^{\frac{\pi}{6}}\left(1-\cos 2\theta\right)d\theta$ $=2\left[\theta-\tfrac{1}{2}\sin 2\theta\right]_{0}^{\frac{\pi}{6}}$ $=2\left[\left(\tfrac{\pi}{6}-\tfrac{1}{2}\sin\tfrac{\pi}{3}\right)-\left(0-\tfrac{1}{2}\sin 0\right)\right]$ $=2\left[\frac{\pi}{6}-\frac{1}{2}\times\frac{\sqrt{3}}{2}\right]$ $=\frac{\pi}{3}-\frac{\sqrt{3}}{2}$ $u=\cos 2\theta$ $du=-2\sin 2\theta d\theta$ $u=1 \Rightarrow \theta=0$ $u=\tfrac{1}{2} \Rightarrow \theta=\tfrac{\pi}{6}$	**4 marks** – Correct solution **3 marks** – Substantially correct solution **2 marks** – Significant progress towards correct solution **1 mark** – Partial (non-trivial) progress towards correct solution

Year 12	Mathematics Extension 2	Assess Task 4 2023 HSC
	Solutions and Marking Guidelines **Question 15**	

Outcomes Addressed in this Question

MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings.

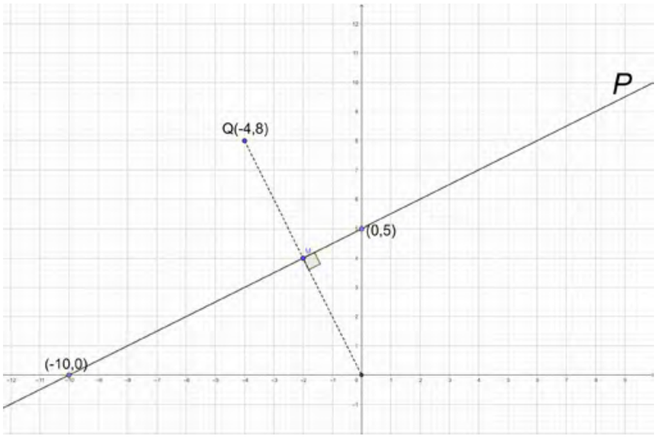
Part	Solutions	Marking Guidelines
(a)	(a) Let a, b be odd integers. i.e. $a = 2k + 1, \quad b = 2j + 1,$ where integers $k, j \geq 0$. Then: $$a^2 + b^3 = (2k+1)^2 + (2j+1)^3$$ $$= 4k^2 + 4k + 1 + 8j^3 + 12j^2 + 6j + 1$$ $$= 2(2k^2 + 2k + 4j^3 + 6j^2 + 3j + 1)$$ Which is an even number for all k, j. Therefore, the statement is false, c can't be odd.	**(a)2 marks**: correct proof. **1 mark**: One element of proof omitted.
(b)	(b) (Part 1) Let the sum of the digits of number *abc* be divisible by 9. i.e. $a + b + c = 9M$ for integer M. The value of $abc = 100a + 10b + c$ $$= 99a + 9b + a + b + c$$ $$= 99a + 9b + 9M$$ $$= 9(11a + b + M)$$ Hence *abc* is divisible by 9. (Part 2) Proving the converse: Let the number *abc* be divisible by 9. i.e. $100a + 10b + c = 9N$ for integer N. $$\therefore\ 99a + 9b + a + b + c = 9N$$ $$a + b + c = 9(N - 11a - b)$$ Hence $a + b + c$ is divisible by 9. Both parts show that the "If and only if" statement is true.	**(b) 3 marks:** Correct proof of iff statement. **2 marks:** Correct proof in only one "direction"; or significant progress in both "directions". **1 mark:** Significant relevant progress.
(c)	(c) (i) Consider the difference: $$(a+b)^2 - 4ab = a^2 + 2ab + b^2 - 4ab$$ $$= a^2 - 2ab + b^2$$ $$= (a-b)^2 \geq 0$$ $$\therefore\ (a+b)^2 \geq 4ab$$ (ii) $x^2 + 3x + 2 = (x+2)(x+1)$ So, from part (i), let $a = x^2 + 3x + 2$, and let $b = \frac{1}{x+1}$. Substituting into the statement in (i): $$\left(x^2 + 3x + 2 + \frac{1}{x+1}\right)^2 \geq 4(x^2 + 3x + 2)\left(\frac{1}{x+1}\right)$$ $$= 4 \times \frac{(x+2)(x+1)}{(x+1)}$$ $$= 4(x+2) \text{ since } x \neq -1$$ $$= 4x + 8$$ $$> 4x \text{ for all values of } x.$$	**(c)(i) 1 mark:** Correct solution. **(ii) 2 marks:** Proper set up and proof, utilising "Hence". **1 mark:** One component of proof incomplete.

(d)	(d) Assume that the log statement is rational, i.e. Assume: $log_x y = \frac{p}{q}$ for $p, q \in \mathbb{Z}^+$ and where ***p* and *q* have no common factors.** $\therefore x^p = y^q$ If x is even and y is odd. $x^p = (2k)^p = 2(2^{p-1})k^p$ Which is even for all integer values of k. However, $y^q = (2j+1)^q = (2j)^q + q(2j)^{q-1} + \cdots .+1$ where every term except for the last will involve a factor of 2. Hence $y^q = (2j+1)^q$ is an odd value. $\therefore x^p \neq y^q$ so $log_x y \neq \frac{p}{q}$ by contradiction. So $log_x y$ is irrational, since the assumption is false.	**(d) 2 marks:** Proper set up and proof, with justification (e.g. explains contradiction). **1 mark:** One component of proof incomplete.
(e)	(e) <u>Step 1:</u> When n =1: $LHS = \frac{d}{dx}(x+1)e^{x-1}$ using the product rule $= (x+1)e^{x-1} + e^{x-1}$. $= (x+2)e^{x-1}$ $RHS = (x+1+1)e^{x-1}$ $= (x+2)e^{x-1} = LHS$ Therefore true when n = 1. <u>Step 2:</u> Assume $\frac{d^k}{dx^k} = \left((x+1)\, e^{x-1}\right) = (x+k+1)\, e^{x-1}$ Prove $\frac{d^{k+1}}{dx^{k+1}} = \left((x+1)\, e^{x-1}\right) = (x+k+2)\, e^{x-1}$ $LHS = \frac{d}{dx}(x+k+1)\, e^{x-1}$ $= (x+k+1)e^{x-1} + e^{x-1}$ $= (x+k+2)e^{x-1}$ $= RHS$ Therefore proven by Mathematical Induction.	**(e) 2 marks:** Both steps correct. **1 mark:** 1 step fully correct.
(f)	(f) <u>Step 1:</u> When x =1, $LHS = \frac{1}{e}$; $RHS = 1$ $\frac{1}{e} < 1$ since $e > 1$. So $LHS < RHS$. Therefore true when x =1. <u>Step 2:</u> Assume $e^{-k} < \frac{1}{k}$ Prove $e^{-(k+1)} < \frac{1}{k+1}$ $LHS = e^{-k}\, e^{-1} < \frac{1}{k} \times \frac{1}{e} = \frac{1}{ke}$ $< \frac{1}{2k}$ since $e > 2$ $\leq \frac{1}{k+1}$ since $k \geq 1$. Therefore $LHS < RHS$ Statement proven by Mathematical Induction.	**(f) 3 marks:** All aspects of induction correct. **2 marks:** One component of proof incomplete. **1 mark:** Significant relevant progress.

Year 12	Mathematics Extension 2	Assess Task 4 2023 HSC
	Solutions and Marking Guidelines **Question 16**	

Outcomes Addressed in this Question

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems.
MEX12-6 uses mechanics to model and solve practical problems.
MEX12-7 uses various mathematical techniques and concepts to model and solve structured, unstructured, and multi-step problems.

Part	Solutions	Marking Guidelines
(a) **MEX12-4**	(a) (i)  (ii) $z+4-8i=\|z\|$ means that the vector representation of the number zQ must be a real number, and hence be horizontal. i.e. the vector $\overrightarrow{QP}$ when *y* = 8 occurs when *P*=(6,8). So $z=6+8i$. <u>Alternative algebraic solution:</u> $x+iy+4-8i=\sqrt{x^2+y^2}$ $\therefore \sqrt{x^2+y^2}-x-iy=4-8i$ Equating real and imaginary parts gives: $y=8;\qquad \sqrt{x^2+y^2}=4+x$ Substitute *y*=8 and square both sides: $x^2+64=16+8x+x^2$ $48=8x$ $6=x\ \rightarrow\ z=6+8i$	**(a)(i) 1 mark:** Correct diagram. **(a) (ii) 2 marks:** Correct solution. **1 mark:** Significant relevant progress.
(b) **MEX12-6**	(b) Particle is at the extremes when *v* = 0, and when substituted into the given equation, we have: $0=21-4x-x^2\ =\ (7+x)(3-x)$ Therefore extremes are at *x*= -7, 3, so the amplitude = 5 units. Alternatively, $v^2=25-(x+2)^2$ So a^2 = 25, giving an amplitude of 5 units.	**(b) 1 mark:** Correct answer.

(c) **MEX12-6**	(c) (i) $x = 5\ \cos\left(3t + \frac{\pi}{4}\right)$ $\dot{x} = -15\ \sin\left(3t + \frac{\pi}{4}\right)$ $\ddot{x} = -45\ \cos\left(3t + \frac{\pi}{4}\right) = -9x$ $\ddot{x} = -n^2x$ where $n = 3$. Therefore, period = $\frac{2\pi}{3}\ s$ (ii) Maximum speed occurs when acceleration = 0. $\therefore \cos\left(3t + \frac{\pi}{4}\right) = 0$ $\left(3t + \frac{\pi}{4}\right) = \frac{\pi}{2}$ $\rightarrow t = \frac{\pi}{12} = 0 \cdot 262\ s.$ The 2nd time it reaches this speed (in the opposite direction) is half of the period later. Second instance: $t = \frac{\pi}{12} + \frac{1}{2}\left(\frac{2\pi}{3}\right) = \frac{5\pi}{12} = 1 \cdot 309s$	**(c)(i) 2 marks:** Correct definition of SHM and period. **1 mark:** One component correct. **(c) (ii) 2 marks:** Complete solution. **1 mark:** One component of solution correct.
(d) **MEX12-4** **MEX12-7**	(d) $z = e^{i\theta} \rightarrow \|z\| = 1$ (i) $z^n - \frac{1}{z^n} = e^{ni\theta} - e^{-ni\theta}$ $= \cos(n\theta) + i\sin(n\theta) - (\cos(-n\theta) + i\sin(-n\theta))$ $= i\sin(n\theta) - i\sin(-n\theta)$ since cos function is even. $= 2i\sin(n\theta)$ since sin function is odd. (ii) Using the binomial expansion, $\left(z - \frac{1}{z}\right)^5 = z^5 - 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z}\right)^2 - 10z^2\left(\frac{1}{z}\right)^3 + 5z\left(\frac{1}{z}\right)^4 - \left(\frac{1}{z}\right)^5$ $= z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}$ $= (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10\ (z - z^{-1})$ which is equal to that required. (iii) Merging statements from parts (i) and (ii): $(2i\sin\theta)^5 = \left(z - \frac{1}{z}\right)^5 = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$ $32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$ $\therefore \sin^5\theta = \frac{1}{32i}(2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta)$ $= \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$ $\therefore \int \sin^5\theta\, d\theta = \frac{1}{16}\int(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)d\theta$ $\frac{1}{16}\left(\frac{-1}{5}\cos 5\theta + \frac{5}{3}\cos 3\theta - 10\cos\theta\right) + c$	**(d) (i) 2 marks:** Satisfies "show" instruction. **1 mark:** One component of proof incomplete. **(d) (ii) 2 marks:** Satisfies "show" instruction. **1 mark:** One component of proof incomplete. **(d) (iii) 3 marks:** Complete solution with working/justificat ion. **2 marks:** One component of required response incomplete. **1 mark:** Significant relevant progress.